Europ. J. Combinatorics (1990) **11**, 473–484

Sums of Cuts and Bipartite Metrics

A. V. KARZANOV

Let m be an integer-valued metric on a finite set V such that the length of any circuit on V is even, and let $H=(VH, EH)$ be an undirected graph with $VH \subseteq V$. A family $\{m_1, \ldots, m_k\}$ of metrics on V is called an H-packing for m if the value $m_1(x, y)+\cdots+m_k(x, y)$ does not exceed $m(x, y)$ for any $x, y \in V$ and equals $m(x, y)$ for each edge $xy \in EH$. A metric m' on V is said to be induced by a graph G if there is a mapping σ from V onto VG such that for any $x, y \in V$, $m'(x, y)$ is equal to the distance in G from $\sigma(x)$ to $\sigma(y)$. It is known that if $|VH| \leqslant 4$ then there exists an H-packing for m consisting of metrics induced by the graph K_2 (i.e. cut metrics), and this is, in general, false when $|VH|>4$.

We prove that if $|VH|=5$ then there exists an H-packing for m consisting of metrics induced either by the graph K_2 or by the graph $K_{2,3}$. Also other results on packings and decompositions of metrics are presented.

1. INTRODUCTION

Throughout this paper, by a *graph* we mean a finite undirected graph without loops and multiple edges; VG is the vertex set and EG is the edge set of a graph G. An edge with end vertices x and y is denoted by xy.

Let V be a finite set of n elements and (V, m) be a semi-metric space, i.e. $m(x, x)=0$, $m(x, y)=m(y, x) \geqslant 0$ and m satisfies the triangle inequalities $m(x, y)+m(y, z) \geqslant m(x, z)$, $x, y, z \in V$. For brevity, we refer to m as a *metric* (rather than a semi-metric) *on* V; $m(x, y)$ will be denoted by $m(xy)$. A metric m is *positive* if $m(xy)>0$ for all distinct $x, y \in V$.

We will identify m with the corresponding function on the edge set EK_V of the complete graph K_V with the vertex set V. Then the set of metrics on V forms a polyhedral cone $\mathcal{M}_V$ in the $\binom{n}{2}$-dimensional euclidean space $\mathbb{R}^{EK_V}$ (the co-ordinates of which correspond to the edges of K_V), called the *metrical cone*. The metrics belonging to extreme rays in $\mathcal{M}_V$ are called *primitive*. The obvious fact is that if $m(xy)=0$ for some $x \neq y$, then m is primitive if the corresponding metric on the set obtained from V by identifying x and y is primitive.

For a connected graph G, we say that a metric m' on V is *induced by* G if there is a mapping σ from V *onto* VG such that $m'(xy)=d^G(\sigma(x)\sigma(y))$ for $x, y \in V$, where $d^G(uv)$ denote the distance in G between vertices u and v (assuming that the length of each edge of G is 1). An elementary example gives a metric m' induced by the graph K_2 (K_p is the complete graph with p vertices). Such m' is often called a *cut metric*. In other words, $m'=\rho X$ is generated by a proper subset X of V as follows:

$$\rho X(xy) := \begin{cases} 1 & \text{if } |\{x, y\} \cap X|=1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that m' is primitive since the metric d^{K_2} is obviously primitive. For a cut metric m' and a real number $t \geqslant 0$, the metric tm' is called *Hamming*.

0195-6698/90/050473 + 12 \$02.00/0

One natural problem on metrics is:

(D): *given a metric m on V, decide whether m is decomposable into a sum*

$$m = m_1 + \cdots + m_k, \tag{2}$$

where $m_1, \ldots, m_k$ are metrics from a certain collection S.

In particular, if S is the set S_1 of Hamming metrics, we obtain the membership problem: decide whether a metric m is contained in the *Hamming* (or *cut*) *cone* $\mathcal{H}_V$. [$\mathcal{H}_V$ is the convex hull of the set of Hamming metrics on V. An equivalent definition: $\mathcal{H}_V$ is the set of metrics on V such that (V, m) is embeddable isometrically into L^1 (see [1]).] Unfortunately, for this 'simplest' collection S_1 the problem (D) already turns out to be NP-hard; this follows from the NP-hardness of the separation problem for $\mathcal{H}_V$ [7] and the fact that the membership problem and the separation problem are polynomially equivalent for a large class of convex sets [6]. It was proved in [3] that the problem is NP-complete if S is the set S_2 of cut metrics. However, a number of non-trivial sufficient conditions on a metric to be decomposable into a sum of metrics in S_1 or S_2 is know. For example, see [4, 5]. One more of them will be pointed out in statement (1.4) below.

Now we introduce a notion closely related to decompositions of metrics. Let G be a connected graph with $VG = V$, the edges $e \in EG$ of which have non-negative real-valued *lengths* $l(e)$, and let H be a graph with $VH \subseteq V$.

Definition. A family $\{m_1, \ldots, m_k\}$ of metrics (possibly repeated) on V is called an *H-packing* for l if

$$l(xy) \geq m_1(xy) + \cdots + m_k(xy) \qquad \text{for all } x, y \in V; \tag{3}$$

and

$$d_l(s, t) = m_1(st) + \cdots + m_k(st) \qquad \text{for all } st \in EH. \tag{4}$$

(Here $d_l(xy)$ denotes the distance between vertices $x, y \in VG$ with respect to l.) In order to demonstrate a relation between packings and decompositions we need the following definition:

Definition. An *extremal graph* of a metric m on V is a minimal (with respect to inclusion) graph H such that for any $u, v \in V$, there is an edge $st \in EH$ satisfying

$$m(su) + m(uv) + m(vt) = m(st). \tag{5}$$

(In [10, 11] the term 'antipode graph' was introduced for such a graph in the case of a positive metric.) For example, if m is a cut metric ρX, then a graph H is extremal for m if it consists of two vertices $x \in X$ and $y \in V - X$ and one edge xy.

Considering (2)–(5) for $G = K_V$ and $l = m$ and applying triangle inequalities we easily obtain the following:

(1.1) *If H is an extremal graph for a metric m and $\mathcal{F} = \{m_1, \ldots, m_k\}$ is an H-packing for m, then $\mathcal{F}$ is a decomposition of m, i.e.* (2) *is valid.*

The problem which will be the focus of the present paper is: for any fixed H, determine a minimal collection S of metrics so that, for any connected graph G with $VG \supseteq VH$ and any length function l on EG, there exists an H-packing for l consisting of metrics in S. Such a problem is related to multicommodity flows, as we explain in

Section 4. In particular, if S is the collection of Hamming metrics, such a relation enables us to derive the following statement from a multicommodity flow theorem of Papernov [12]:

(1.2) *If H is K_4 or C_5 or a union of two stars, then there exists an H-packing for l consisting of Hamming metrics.*

(A *star* is a connected graph, the edges of which have a common vertex; C_5 is the circuit with five vertices. It is easy to see that any proper subgraph of K_4 or C_5 having no isolated vertices is a union of two stars.) Statement (1.2) cannot be strengthened in term of the graphs H; more precise, one can prove that if H is not as in (1.2) and it contains no isolated vertex, then for any $V \supseteq VH$ there exists a metric m on V having no H-packing of Hamming metrics. There is a stronger, 'half-integral', version of (1.2). We say that a (non-negative) function l is *cyclically even* if it is integer-valued and each circuit in G has an even length, i.e. $l(x_0x_1) + \cdots + l(x_{r-1}x_r) + l(x_rx_0)$ is even for any $x_0x_1, \ldots, x_{r-1}x_r, x_rx_0 \in EG$.

(1.3)[7] *If H is as in* (1.2) *and l is cyclically even, then there exists an H-packing for l consisting of cut metrics.*

(Another, simpler, proof of (1.3) is given in [13].) (1.1) and (1.3) imply the following:

(1.4) *If a metric m has K_4 or C_5 or a union of two stars as an extremal graph, then $m \in \mathscr{H}_V$. If, in addition, m is cyclically even, then m is decomposable into a sum of cut metrics.*

A simplest example of a metric not decomposable into a sum of cut metrics gives any metric $d^{K_{p,q}}$ for $p \geqslant 2$ and $q \geqslant 3$ ($K_{p,q}$ is the complete bipartite graph with parts of p and q vertices). It is known that such a metric is primitive; see, for example, [10, 2]. The metric induced by the graph $K_{p,q}$ is called a *p,q-metric.*

The main result of the present paper is the following:

THEOREM 1. *If l is a cyclically even function on the edges of a connected graph G, H is a graph with $VH \subseteq VG$, and $|VH| = 5$, then there exists an H-packing for l consisting of cut metrics and* 2,3*-metrics.*

(1.5) (Corollary from (1.1) and Theorem 1) *Every cyclically even metric having an extremal graph H with $|VH| = 5$ is representable as a sum of cut metrics and* 2,3*-metrics.*

Theorem 1 will be proved in Section 2; the proof will provide a strongly polynomial algorithm for finding a required packing. Note that, in fact, a slightly stronger version of this theorem will be proved in which one asserts that all used 2,3-metrics can be chosen to coincide on the set VH. In particular, if m is a cyclically even metric on a set V of five elements, then m is a sum of cut metrics and of some number of copies of one 2,3-metric on V.

Theorem 1 can be reformulated in polyhedral terms as follows (a corresponding statement can be stated also for (1.3)). For $U \subset EK_V$, define the cone $\mathscr{M}_{V,U}$ to be the non-negative linear hull of the cone $\mathscr{M}_V$ and the vectors l_e, $e \in EK_V - U$, where l_e is the eth unit basis vector in $\mathbb{R}^{EK_V}$ (such a cone occurred in [10, 11]).

(1.6) *Let the edges in U span exactly 5 vertices. Then*:
(i) *each extreme ray of $\mathcal{M}_{V,U}$ is $\{\lambda a: \lambda \geq 0\}$, where a is either I_e ($e \in EK_V - U$) or a cut metric or a* 2,3-*metric*;
(ii) *if l is an integral vector in $\mathcal{M}_{V,U}$ such that $l(xy)+l(yz)+l(xz)$ is even for any distinct $x, y, z \in V$, then l is a sum of integral vectors lying on extreme rays in $\mathcal{M}_{V,U}$.*

Section 3 contains some generalizations. By the *metrical spectrum $MS(H)$* of a graph H we shall mean the minimal set S so that: (i) each $d \in S$ is an integer-valued metric on a set $V(d)$, and td is not integer-valued for $0<t<1$; and (ii) for any connected graph G with $VG \supseteq VH$ and a function l: $EG \to \mathbb{R}_+$, there exist metrics $d_1, \ldots, d_k \in S$ and reals $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\{\lambda_1 m_1, \ldots, \lambda_k m_k\}$ is an H-packing for l, where m_i is a metric on VG induced by d_i, i.e. $m_i(xy)=d_i(\sigma(x)\sigma(y))$, $x, y \in V$, for some mapping σ from VG onto $V(d_i)$. For example, the metrical spectrum of K_4 consists uniquely of the metric d^{K_2} (by (1.2)), and the metrical spectrum of K_5 consists of the two metrics d^{K_2} and $d^{K_{2,3}}$ (by Theorem 1). In Section 3 we give a complete description of the set of graphs H for which $MS(H)$ is finite.

2. Proof of Theorem 1

Put $T:=VH$ and $E:=EK_V$. It suffices to prove that if $|T| \leq 5$ and m is a cyclically even function (not necessarily metric) on E, then there exists a K_T-packing for m consisting of cut metrics and 2,3-metrics. Our method of proof uses ideas of [8], developed there for proving that if m' is a primitive metric having an extremal graph H' with $|VH'|=5$, then m' is proportional to a 2,3-metric. In particular, the statements (2.1), (2.3) and (2.4) occurred in [8]; we give their proofs here in order to make our description self-contained.

By a *path*, or *x–y path*, on V we mean a sequence $P=x_0x_1\cdots x_k$ of distinct elements $x=x_0, x_1, \ldots, x_k=y$ of V. $e_i=x_ix_{i+1}$ is an *edge* of P and the value $m(P)=\sum(m(e_i): i=1, \ldots, k)$ is the *length* of P (with respect to m); P is *shortest* if $m(P)=d_m(xy)$.

For $xy \in E$ and $u, v, w \in V$ put:

$$\varphi(xy)=\varphi_m(xy):=\min\{d_m(sx)+d_m(xy)+d_m(yt)-d_m(st): s, t \in T\};$$
$$\Delta(u, v, w)=\Delta_m(u, v, w):=d_m(uv)+d_m(vw)-d_m(uw).$$

The cyclically evenness of m implies that $\varphi(xy)$ and $\Delta(u, v, w)$ are even.

We proceed by induction on

$$\alpha=\alpha(V, T, m):=|V|+|\{e \in E: m(e)>0\}|+|\{e \in E: \varphi(e)>0\}|$$
$$+|\{(s, x, t): x \in V, s, t \in T, \Delta(s, x, t)>0\}|.$$

By (1.3), the theorem is true if $|T| \leq 4$.

First of all we show that one may consider only the case when the following hold:

$$m \text{ is a positive metric;} \tag{6}$$

$$\text{each edge } e \in E \text{ is contained in some shortest } s\text{–}t \text{ path, } s, t \in T, \text{ i.e. } \varphi(e)=0; \tag{7}$$

$$\text{each } p \in T \text{ is contained in some shortest } s\text{–}t \text{ path with } s, t \in T-\{p\}. \tag{8}$$

This is achieved by use of the following simple reductions.

(i) Suppose that $m(xy)=0$ for some $xy \in E$. Identify x and y with a new vertex z obtaining corresponding V', T'. For $u, v \in V'$, define $m'(uv) := m(uv)$ if $u, v \neq z$, and $m'(uv) := \min\{m(ux), m(uy)\}$ if $v = z$. Then m' is cyclically even and $\alpha(V', T', m') < \alpha(V, T, m)$. By induction there exists a required $K_{T'}$-packing for m' which naturally determines a required K_T-packing for m.

Thus, one can assume that $m(e) > 0$ for all $e \in E$.

(ii) Suppose that $m(e) \geq 2$ and $\varphi(e) > 0$ for some $e \in E$. Let a be the maximum even number not exceeding $\max\{m(e), \varphi(e)\}$. Put $m'(e) := m(e) - a$ and $m'(e') := m(e')$, $e' \in E - \{e\}$. Clearly, m' is cyclically even, $d_{m'}(st) = d_m(st)$ for all $s, t \in T$, and $\alpha(V, T, m') \leq \alpha(V, T, m)$. Three cases are possible: (a) $m'(e) = 0$; (b) $\varphi_{m'}(e) = 0$; and (c) $m'(e) = 1$. In cases (a) and (b), we have $\alpha(V, T, m') < \alpha(V, T, m)$, and the result follows by induction. So we may assume that $\varphi(e) = 0$ for all $e \in E$ with $m(e) \geq 2$. Suppose that there is an edge $xy \in E$ with $m(xy) = 1$. Take $z \in V - \{x, y\}$, and for definiteness let $m(xz) \geq m(yz)$. Since $m(xy) + m(xz) + m(yz)$ is even and m is positive, we have $m(xz) \geq 2$ and $m(xy) + m(yz) = m(xz)$. Since $m(xz) \geq 2$, $\varphi(xz) = 0$, and hence there exists a shortest path $s \ldots xz \ldots t$ with $s, t \in T$. Then the path $s \ldots xyz \ldots t$ is also shortest.

Thus, one can assume that (7) holds. It follows easily from (7) that m is a metric, whence (6) holds.

(iii) Suppose that $\omega(p) > 0$ for some $p \in T$, where $\omega(p) := \min\{\Delta(s, p, t): s, t \in T - \{p\}\}$. Put $X := \{p\}$ and $m' := m - a\rho X$, where

$$a := \min\{\omega(p)/2, \min\{m(py): y \in V - \{p\}\}\}.$$

Since a is an integer, m' is cyclically even. Obviously, $d_{m'}(st) = d_m(st)$ and $d_{m'}(sp) = d_m(sp) - a$ for $s, t \in T - \{p\}$. Furthermore, at least one of the following is true: $\Delta_{m'}(s, p, t) = 0$ for some $s, t \in T - \{p\}$, or $m'(x,y) = 0$ for some $y \in V - \{p\}$, whence $\alpha(V, T, m') < \alpha(V, T, m)$. By induction there exists a required packing for m'. Adding to it a copies of the cut metric ρX we obtain the required packing for m.

Thus, one can assume that (8) holds.

Let H' be the extremal graph for m (it is easy to show that a positive metric has a unique extremal graph). Let $U := EH'$. (7) and (8) imply that

$$\text{each edge } e \in E \text{ is in some shortest } s\text{–}t \text{ path for } st \in U; \tag{9}$$

$$\text{for each } p \in T, \text{ there exists } st \in U \text{ such that } s, t \neq p \text{ and } m(sp) + m(pt) = m(st). \tag{10}$$

In view of (1.3), one can assume that H' is different from K_4, C_5 and a union of two stars. In particular, this implies that $|VH'| = 5$, i.e. $VH' = T$. Also one can show that there are three vertices in H', say s_1, s_2, s_3, such that $s_is_j \in U$, $1 \leq i < j \leq 3$ (otherwise H' is either C_5 or a union of two stars). Let $T_1 := \{s_1, s_2, s_3\}$ and $T_2 := T - T_1 =: \{s_4, s_5\}$.

(2.1) (i) $U = \{s_1s_2, s_2s_3, s_3s_4, s_4s_5\}$;
(ii) $m(s_is_4) + m(s_is_5) = m(s_4s_5)$ for $i = 1, 2, 3$.

PROOF. Consider $p \in T_1$. Let s and t be vertices as in (10). The minimality of the extremal graph H' implies $ps, pt \notin U$. Thus, $\{s, t\} \cap T_1 = \varnothing$ i.e. $\{s, t\} = \{s_4, s_5\}$. Hence, $s_4s_5 \in U$, $ps_4, ps_5 \notin U$ and (ii) is true. □

For $\varnothing \neq X \subset V$, let δX denote the set with one end in X and the other in $V - X$ (a *cut* on V); δX *separates* vertices x and y if $|\{x, y\} \cap X| = 1$. As before, for a metric m

one assumes by definition $m(xx)=0$, $x \in V$. For $x, y \in V$, let $N(x, y)$ denote the set of vertices contained in shortest x–y paths.

(2.2) *Let* $s \in T_1$, $t \in T_2$, $\{p, q\} = T_1 - \{s\}$, *and let* $N(s, t) \cap N(p, q) = \varnothing$. *Put* $X := N(s, t)$,

$$a := \tfrac{1}{2} \min\{\min\{m(sz)+m(zt)-m(st): z \in V - X\},\ \min\{m(pz)+m(zq)-m(pq): z \in X\}\},$$

and $m' := m - a\rho X$. *Then* $m' \geq 0$, *a is an integer* ≥ 1, *and for each* $uv \in U$ *the following holds*:

$$\begin{aligned} d_{m'}(uv) &= m(uv) - a && \text{if } \delta X \text{ separates } u \text{ and } v, \\ &= m(uv) && \text{otherwise.} \end{aligned} \tag{11}$$

Proof. The condition that $N(s, t)$ and $N(p, q)$ are disjoint and the cyclically evenness of m imply that a is an integer ≥ 1. Next, the vertex r in T_2 different from t cannot be in X, by the minimality of H'.

(i) Consider vertices $x, y \in X$, and for definiteness let $m(sx) \leq m(sy)$. We assert that the path $sxyt$ is shortest (for m). This is true for $x = y$ by the definition of $N(s, t)$. Let $x \neq y$. By (9), there is a shortest path $s'xyt'$ for some $s't' \in U$. Since x belongs to no shortest p–q path, we have $s't' \neq pq$. Therefore, $s't' \in \{sp, sq, tr\}$. Suppose $s't' = sp$. It follows from $m(sx) \leq m(sy)$ that $s' = s$. Since the paths syt and $s'xy$ are shortest, the path $sxyt$ is also shortest, as required. The cases $s't' = sq, tr$ are considered analogously.

(ii) Let $x, y \in X$ and $z \in V - X$, and for definiteness let the path $sxyt$ be shortest. Then

$$\begin{aligned} m(xz)+m(zy)-m(xy) &= m(sx)+m(xz)+m(zy)+m(yt)-m(st) \\ &\leq m(sz)+m(zt)-m(st) \geq 2a, \end{aligned} \tag{12}$$

by the definition of a. If $x = y$, we obtain from (12) that $m(xz) \geq a$, whence $m' \geq 0$.

(iii) Let $uv \in U$. Consider a u–v path $P = x_0x_1 \cdots x_k$ shortest for m'. One must prove that $m'(P)$ is equal to $m(uv)$ if $uv = pq$ and equal to $m(uv) - a$ if $uv = sp, sq, tr$. One may assume that k is minimum (by fixed uv). The assertion is obvious when $k = 1$. Let $k \geq 2$. For $i < k-1$, let P_i denote the path $x_0x_1 \cdots x_ix_{i+2} \cdots x_k$; then $m'(P_i) > m'(P)$. We observe that P has the following properties.

(a) $x_ix_{i+1} \in \delta X$. Indeed, suppose that $x_i, x_{i+1} \in X$, and for definiteness let $i \leq k-2$. If $x_{i+2} \in X$, then $m'(e) = m(e)$ for $e = x_ix_{i+1}, x_ix_{i+2}, x_{i+1}x_{i+2}$, and if $x_{i+2} \in V - X$, then $m'(x_ix_{i+1}) = m(x_ix_{i+1})$ and $m'(e) = m(e) - a$ for $e = x_ix_{i+2}, x_{i+1}x_{i+2}$. In both cases, we obtain from $m(x_ix_{i+1}) + m(x_{i+1}x_{i+2}) \geq m(x_ix_{i+2})$ that $m'(P_i) \leq m'(P)$; a contradiction with the minimality of k. The case $x_i, x_{i+1} \in V - X$ is considered similarly.

(b) There is no i such that $x_i, x_{i+2} \in X$, $x_{i+1} \in V - X$. Otherwise, taking (12) into account, we have

$$\begin{aligned} m'(x_ix_{i+2}) = m(x_ix_{i+2}) &\leq m(x_ix_{i+1}) + m(x_{i+1}x_{i+2}) - 2a \\ &= m'(x_ix_{i+1}) + m'(x_{i+1}x_{i+2}), \end{aligned}$$

whence $m'(P_i) \leq m'(P)$; a contradiction.

It follows from (a) and (b) that $k = 2$, $x_0, x_2 \in V - X$ and $x_1 \in X$. Then $uv = pq$, and now we conclude from the definition of a that $m'(P) = m(px_1) + m(x_1q) - 2a \geq m(pq)$. □

Suppose that s, t, a, X and m' are so as in (2.2). It follows easily from (11) that $\varphi_{m'}(e) \leq \varphi_m(e)$ and $\Delta_{m'}(s', x, t') \leq \Delta_m(s', x, t')$ for all $e \in E$, $x \in V$ and $s', t' \in T$. Furthermore, m' is cyclically even (as a is an integer) and $\Delta_{m'}(s', z, t') = 0$ for $s't' = st$ and some $z \in V - X$ or for $s't' = pq$ and some $z \in X$. Therefore, $\alpha(V, T, m') < \alpha(V, T, m)$, and by induction there exists a required packing for m'. Adding to it a copies of the cut metric ρX we obtain a required packing for m.

So we may assume that $N(p, q) \cap N(s, t) \neq \varnothing$ whenever $\{p, q, s\} = T_1$ and $t \in T_2$. Put $a_{ij} := m(s_i s_j)$ for $i = 1, 2, 3, j = 4, 5$; $b_{ij} = b_{ji} := m(s_i s_j)$ for $1 \leq i < j \leq 3$; and $c := m(s_4 s_5)$.

(2.3) *Let $\lambda := a_{14}$. Then all a_{ij} are equal to λ and $b_{12} = b_{23} = b_{31} = c = 2\lambda$.*

PROOF. Let $i \in \{1, 2, 3\}$ and $j \in \{4, 5\}$. Choose a vertex x in $N(s_i, s_j)$ contained in a shortest s_k–s_r path for $\{i, k, r\} = \{1, 2, 3\}$. Then

$$a_{ij} + b_{kr} = m(s_i x) + m(s_j x) + m(s_k x) + m(s_r x) \geq a_{kj} + b_{ir}.$$

Therefore, $a_{1j} + b_{23} = a_{2j} + b_{13} = a_{3j} + b_{12}$. Considering these equalities for $j = 4, 5$ and the equalities in (2.1)(ii) we obtain $a_{1j} = a_{2j} = a_{3j} =: a_j$ and $b_{12} = b_{23} = b_{31} =: b$. Now since s_j is in a shortest s_i–s_k path for some $1 \leq i < j \leq 3$ (by (10)), we have $2a_j = b$, whence $a_4 = a_5 = \lambda$ and $b = c = 2\lambda$. □

(2.3) shows that the restriction of m on T is a metric proportional to $d^{K_{2,3}}$. Now our end is to show that for a metric with such a property there exists a packing consisting of 2, 3-metrics.

For $i, j = 1, \ldots, 5$, put $N_{ij} := N(s_i, s_j)$. Define the sets:

$$S_4 := \{s_4\};$$
$$S_i := N_{i4} - \{s_4\}, \qquad i = 1, 2, 3;$$
$$S_5 := V - (S_1 \cup S_2 \cup S_3 \cup S_4).$$

Clearly, $s_i \in S_i$, $i = 1, 2, 3, 4$. Also (2.3) implies that $s_5 \in S_5$. Below we shall prove the following:

(2.4) *The sets $S_1, \ldots, S_5$ are disjoint.*

The partition $\mathcal{P} := \{S_1, \ldots, S_5\}$ of V defines the 2,3-metric d on V as

$$\begin{aligned} d(xy) &:= 1 \quad \text{if } xy \in (S_i, S_j), \quad i = 1, 2, 3,\ j = 4, 5; \\ &:= 2 \quad \text{if } xy \in (S_1, S_2), (S_2, S_3), (S_3, S_1), (S_4, S_5); \\ &:= 0 \quad \text{otherwise}, \end{aligned}$$

where (X, Y) is the set of edges with one end in X and the other in Y. By (2.3), $m(st) = \lambda d(st)$, $s, t \in T$. Put

$$\beta := \min\{m(s_4 x) : x \in S_1 \cup S_2 \cup S_3\};$$
$$\gamma := \tfrac{1}{2} \min\{m(s_4 x) + m(x s_i) - m(s_4 s_i) : x \in S_5,\ i = 1, 2, 3\}$$

and $a := \min\{\beta, \gamma\}$. Then a is an integer ≥ 1, as follows directly from the definition of $S_1, \ldots, S_k$. Put $m' := m - ad$. Below we shall prove the following:

(2.5) $m' \geq 0$ and $d_{m'}(st) = m(st) - ad(st)$ for all $st \in U$.

In the assumption that (2.4) and (2.5) are true, the proof of Theorem 1 is completed as follows. It is easy to check that the metric $d^{K_{2,3}}$ is cyclically even, whence, in view of integrity of a, the metric m' is cyclically even. It follows from the definition of a that $\alpha(V, T, m') < \alpha(V, T, m)$. By induction there exists a required packing for m'. Adding to it a copies of the metric d yields a required packing for m.

PROOF OF (2.4). It suffices to prove that $N_{i4} \cap N_{j4} = \{s_4\}$ for $1 \leqslant i < j \leqslant 3$. Suppose that $N_{i4} \cap N_{j4}$ contains a vertex x different from s_4. Since $m(s_4x) > 0$ and $m(s_ix) + m(xs_4) = \lambda$ (where λ is defined as in (2.3)), $m(s_ix) < \lambda$; similarly, $m(s_jx) < \lambda$. Hence, $m(s_is_j) < 2\lambda$ contrary to (2.3). □

PROOF OF (2.5). First of all we make several preliminary observations:

$$m'(xy) = m'(xs_4) + m'(s_4y) \qquad \text{for } x \in S_i,\ y \in S_j,\ 1 \leqslant i < j \leqslant 3; \tag{13}$$

$$m'(xz) + m'(zy) \geqslant m'(xs_4) + m'(s_4y) \qquad \text{for } x \in S_i,\ y \in S_j,\ 1 \leqslant i < j \leqslant 3,\ z \in S_5. \tag{14}$$

Indeed, since $m(s_ix) + m(xs_4) = \lambda = m(s_jy) + m(ys_4)$ and $m(s_is_j) = 2\lambda$, the path $s_ixs_4ys_j$ is shortest for m, whence $m(xy) = m(xs_4) + m(s_4y)$ and $m(xz) + m(zy) \geqslant m(xs_4) + m(s_4y)$. Now (13) and (14) follow from the fact that the value $m(e) - m'(e)$ is a for $e \in (S_k, S_r)$, $k = 1, 2, 3$, $r = 4, 5$, and $2a$ for $e \in (S_i, S_j)$.

$$m'(xz) + m'(zy) \geqslant m'(xy) \qquad \text{for } x \in S_i \cup S_4,\ y \in S_i,\ 1 \leqslant i \leqslant 3, \text{ and } z \in S_5 \tag{15}$$

(if $x = y$ then $m'(xy) := 0$). Indeed, as it was shown earlier (see (12)), $\Delta(x, z, y) \geqslant \Delta(s_4, z, s_i)$, where $\Delta(x', z', y')$ is $m(x'z') + m(z'y') - m(x'y')$. Thus, $\Delta(s_4, z, s_i) \geqslant 2a$ (by the definition of a) implies $\Delta(x, z, y) \geqslant 2a$. If $x \in S_i$ then $m'(xy) = m(xy)$ and $m'(e) = m(e) - a$ for $e = xz, xy$, and if $x = s_4$ then $m'(xz) = m(xz) - 2a$ and $m'(e) = m(e) - a$ for $e = xy, zy$, whence (15) follows.

$$m'(s_4x) \geqslant 0 \qquad \text{for } x \in S_5. \tag{16}$$

Indeed, for definiteness let $m(s_1x) \leqslant m(s_2x) \leqslant m(s_3x)$. Suppose that $m(s_1x) + m(s_2x) > 2\lambda$. Then $m(s_2x) > \lambda$ and the edge s_2x belongs to no shortest s–t paths for $st = s_1s_2, s_2s_3, s_3s_1$. Hence, s_2x is in a shortest s_4–s_5 path (by (9)). But $m(s_4s_2) = m(s_2s_5) = \lambda$; a contradiction. Thus, $m(s_1x) + m(s_2x) = 2\lambda$. Now we obtain from $\Delta(s_4, x, s_i) \geqslant 2a$, $i = 1, 2$, and $m(s_1s_4) + m(s_4s_2) = 2\lambda$ that $m(s_4x) \geqslant 2a$, whence (16) follows.

We show $m'(uv) \geqslant 0$ for all u, v. Let $u \in S_i$, $v \in S_j$. If $i = j$ then $m'(uv) = m(uv) \geqslant 0$. If $i \neq j$, $m'(uv) \geqslant 0$ follows for $i = 4, j = 5$ from (16), and for $i = 4, j = 1, 2, 3$ from the definition of a. Therefore, $m'(uv) \geqslant 0$ for $\{i, j\} = \{1, 2\}, \{2, 3\}, \{3, 1\}$ by (13). Finally, this follows for $i = 1, 2, 3$, $j = 5$ from (15) (putting $x = y = u$ and $z = v$).

We finally show the second half of (2.5). Let $st \in U$ be fixed. Consider an s–t path $P = x_0x_1 \cdots x_k$ shortest for m'. One must prove that $m'(P) = m(st) - ad(st)$. We may assume that P satisfies the following conditions: (i) the number of vertices x_i different from s_4 is minimum; and (ii) the number of indices i such that $x_i = s_4$ is maximum subject to (i). Let P_i denote the path $x_0x_1 \cdots x_ix_{i+2} \cdots x_k$. We observe that P satisfies the following properties:

(a) x_i and x_{i+1} belong to different sets in $\mathscr{P}$. Indeed, if, say, $x_i, x_{i+1} \in S_k$, $x_{i+2} \in S_r$, $k \neq r$, then $x_{i+1} \neq s_4$ and we have from $m'(x_ix_{i+1}) = m(x_ix_{i+1})$ and $m'(e) = m(e) - ad(s_ks_r)$ for $e = x_{i+1}x_{i+2}, x_ix_{i+2}$ that $m'(P_i) \leqslant m'(P)$, which contradicts (i).

(b) There is no i such that $x_i \in S_k$ and $x_{i+1} \in S_r$ for $k, r \in \{1, 2, 3\}$, $k \neq r$. Otherwise the path $x_0x_1 \cdots x_is_4x_{i+1} \cdots x_k$ is shortest for m' (by (13)), contrary to (ii).

(c) There is no i such that $x_i \in S_k \cup S_4$, $x_{i+1} \in S_5$ and $x_{i+2} \in S_r \cup S_4$ for $k, r \in \{1, 2, 3\}$. Otherwise, $m'(P_i) \leqslant m'(P)$ (by (14) or (15)), contrary to (i).

(d) There is no i such that $x_{i+1} \in S_k$ for $k \in \{1, 2, 3\}$ and either $x_i = s_4$ and $x_{i+2} \in S_5$ or

$x_i \in S_5$ and $x_{i+2} = s_4$. Otherwise, it follows from $m'(e) = m(e) - a$ for $e = x_ix_{i+1}$, $x_{i+1}x_{i+2}$ and from $m'(x_ix_{i+2}) = m(x_ix_{i+2}) - 2a$ that $m'(P_i) \leq m'(P)$, contrary to (i).

Now in the case $s = s_4$ and $t = s_5$, we conclude easily from (a)–(d) that $P = s_4s_5$. Then $m'(P) = m'(s_4s_5) = m(s_4s_5) - 2a$. In the case $s = s_k$ and $t = s_r$, $1 \leq k < r \leq 3$, we conclude from (a)–(d) that either $P = s_ks_r$ or $P = s_ks_4s_r$, whence $m'(P) = m(s_ks_r) - 2a$. □

This completes the proof of Theorem 1.

In fact, the proof of the theorem contains an algorithm for finding a required packing the running time of which is a polynomial in $|V|$.

Remark. One can see that if m and m' are functions figured in (2.5), then the metrics m and $d_{m'}$ are proportional on EK_T. Furthermore, obviously, $d_{m'}$ satisfies the properties as in (9) and (10). This gives the strengthening of Theorem 1 pointed out in the Introduction.

3. Metrical Spectra of Graphs

Theorem 1 has the following corollary:

(3.1) *Let H be a union of K_3 and a star, and let l be a cyclically even function on the edges of a connected graph G with $VG \supseteq VK$. Then there exists an H-packing for l consisting of cut metrics and* 2, 3*-metrics.*

Proof. For definiteness let H have the edges s_1s_2, s_2s_3, s_3s_1 and st_i, $i = 1, \ldots, r$ (possibly vertices in $\{s, t_1, \ldots, t_r\}$ coincide with some vertices in $\{s_1, s_2, s_3\}$). Add to G a new vertex t and the edges $t_1t, \ldots, t_rt$ forming the graph G'. Let l' be a cyclically even function on EG' such that $l'(e) = l(e)$ for $e \in EG$ and

$$d_{l'}(pq) = d_l(pq) \qquad \text{for all } pq \in EH; \tag{17}$$

$$d_{l'}(st) = d_l(st_i) + l'(t_it) \qquad \text{for } i = 1, \ldots, r; \tag{18}$$

it is easy to show that such a function exists. Let T be the set of different vertices among s_1, s_2, s_3, s, t; then $|T| \leq 5$. By Theorem 1 there is a K_T-packing $\{m'_1, \ldots, m'_k\}$ for l' consisting of cut metrics and 2, 3-metrics. Let m_j be the restriction of m'_j on VG. One can see that each m_j is a cut metric or a 2, 3-metric or a sum of cut metrics. It obviously follows from (17) and (18) that $m_1, \ldots, m_k$ determine a required H-packing for l. □

(1.2), (3.1) and Theorem 1 give the metrical spectrum $MS(H)$ when $|VH| \leq 5$ or H is a union of two stars or H is a union of K_3 and a star; in these cases $MS(H)$ is either $\{d^{K_2}\}$ or $\{d^{K_2}, d^{K_{2,3}}\}$. Now we study the metrical spectra for the other graphs H. One may assume that H has no isolated vertices. A direct check-up shows that H is one of the following:

$$H \text{ has a matching of three edges;} \tag{19}$$

$$H \text{ consists of two disjoint graphs } K_3. \tag{20}$$

We assert that the set $MS(H)$ is infinite for any graph H as in (19), and that it is finite for the graph H as in (20).

Clearly, if H'' is a subgraph of a graph H', then $MS(H'') \subseteq MS(H')$. Let H_0 be the graph consisting of three disjoint edges. Thus in order to prove the infiniteness of $MS(H)$

for H as in (19) it suffices to show that the infiniteness of $MS(H_0)$. To see the latter, take positive integers $p, q, r \geq 2$, and let G be the graph the vertices of which are the triples (i, j, k), $i=1, \ldots, p$, $j=1, \ldots, q$, $k=1, \ldots, r$, and the edges of which are the pairs $\{(i, j, k), (i', j', k')\}$ such that either $|i-i'|+|j-j'|+|k-k'|=1$ or $i-i'=j-j'=k-k'=1$. Then the metric d^G is primitive and the edges of its extremal graph correspond to the pairs $\{(p, 1, 1), (1, q, r)\}$, $\{(1, q, 1), (p, 1, r)\}$ and $\{(1, 1, r), (p, q, 1)\}$ (see [8]). So d^G belongs to $MS(H_0)$. Hence $MS(H_0)$ is infinite.

Now consider the graph H as in (20); for definiteness let $VH=\{s_1, \ldots, s_6\}$ and $EH=\{s_is_j: 1 \leq i<j \leq 3 \text{ or } 4 \leq i<j \leq 6\}$. In order to prove that $MS(H)$ is finite take a connected graph G with $VG \supseteq VH$ and a non-negative function l on EG. Add to G new vertices $t_1, \ldots, t_6$ and the edges s_it_i, $i=1, \ldots, 6$, forming the graph G'. Let H' be the graph with the vertex set $\{t_1, \ldots, t_6\}$ and the edge set $\{t_it_j: 1 \leq i<j \leq 3 \text{ or } 4 \leq i<j \leq 6\}$; then H' is isomorphic to H. Take $\tau>0$ such that $\tau d_l(pq) \leq 1/2$ for all $pq \in EH$. For $i=1, \ldots, 6$, let $J_i:=\{j: s_is_j \in EH\}$. Define the function l' on EG' by

$$\begin{aligned} l'(e) &:= \tau l(e) \qquad \text{for } e \in EG, \\ &:= \tfrac{1}{2}(1+\tau d_l(s_js_k)-\tau d_l(s_is_j)-\tau d_l(s_is_k)) \qquad \text{for } e=s_it_i, \\ &\qquad\qquad i=1, \ldots, 6, \{j, k\}=J_i. \end{aligned}$$

Then $l' \geq 0$. It is easy to check that

$$d_{l'}(pq)=\tau d_l(pq) \qquad \text{for all } pq \in EH; \tag{21}$$

$$d_{l'}(p'q')=1 \qquad \text{for all } p'q' \in EH'. \tag{22}$$

Introduce the metric d^Γ of distances in the following special graph Γ (this metric occurred in [8, 9]). Γ consists of 16 vertices $p_1, \ldots, p_6$, x_{ij} $(1 \leq i \leq 3, 4 \leq j \leq 6)$ and v and of 27 edges p_ix_{ij}, p_jx_{ij} and $x_{ij}v$, $1 \leq i \leq 3$, $4 \leq j \leq 6$. In [9] the following statement (that is a particular case of a general theorem) was proved: if H' is the graph as above and the function l' satisfies (22), then there is a mapping $\sigma: VG' \to V\Gamma$ such that $\sigma(t_i)=p_i$, $i=1, \ldots, 6$, and the metric m^σ H'-decomposes l'. Here $m^\sigma(xy):=d^\Gamma(\sigma(x)\sigma(y))$, $x, y \in VG'$, and we say that m' H'-decomposes l' if $l'(e)-\lambda m'(e) \geq 0$ for all $e \in EG'$ and $d_{l'-\lambda m'}(pq)=d_{l'}(pq)-\lambda m'(pq)$ for all $pq \in EH'$ and some $\lambda>0$. This easily implies that for some finite k there exists an H'-packing $\{\lambda_1 m^{\sigma_1}, \ldots, \lambda_k m^{\sigma_k}\}$, where σ_i is some mapping as above. Then, by (21), $\{m_1, \ldots, m_k\}$ is an H-packing for l, where m_i is the restriction of the metric $\tau\lambda_i m^{\sigma_i}$ on VG. Thus, the cardinality of $MS(H)$ does not exceed the number of primitive metrics, each being a restriction of the metric d^Γ on a subset of $V\Gamma$. Therefore $MS(H)$ is finite.

The conjecture is: if H is as in (20) and l is cyclically even, then there exists a H-packing $\{\lambda_1 m_1, \ldots, \lambda_k m_k\}$ for l such that $m_i=m^{\sigma_i}$ for some $\sigma_i: VG \to V\Gamma$ and all λ_i are multiples of $\frac{1}{2}$.

4. Relation of H-packings to Multicommodity Flows

Consider a connected graph G, a graph H with $VH \supseteq VG$, and functions $c: EG \to \mathbb{R}_+$ (a *capacity* function) and $g: EH \to \mathbb{R}_+$ (a *demand* function). For $st \in EH$, denote by $\mathcal{P}_{st}$ the set of simple paths in G connecting s and t, and let $\mathcal{P}:=\bigcup(\mathcal{P}_{st}: st \in EH)$. The multicommodity flow problem $F(G, H, c, g)$ (in the so-called 'edge-path' form) is: find a function (*multicommodity flow*) $f: \mathcal{P} \to \mathbb{R}_+$ so that:

$$\sum (f(P): P \in \mathcal{P}, e \in P) \leq c(e) \qquad \text{for } e \in EG; \tag{23}$$

and

$$\sum (f(P): P \in \mathcal{P}_{st}) = g(st) \qquad \text{for } st \in EH; \tag{24}$$

or establish that such a function does not exist.

By Farkas' lemma, (23)–(24) is solvable iff, for any $l: EG \to \mathbb{R}_+$ and $b: EH \to \mathbb{R}$, $c \cdot l \geq g \cdot b$ holds provided that

$$\sum (l(e): e \in P) \geq b(st) \qquad \text{for } P \in \mathcal{P}_{st},\ st \in EH, \tag{25}$$

where $a \cdot a'$ denotes the inner product of a and a'. (25) is equivalent to $b(st) \leq d_l(st)$ for $st \in EH$. So we have the following:

(4.1) *$F(G, H, c, g)$ is solvable (i.e. a required f exists) iff*

$$c \cdot l \geq \sum (g(st)d_l(st): st \in EH) \tag{26}$$

holds for any $l: EG \to \mathbb{R}_+$.

Now suppose we know that there is a set $S = \{m_1, \ldots, m_N\}$ of metrics on VG such that for any $l: EG \to \mathbb{R}_+$ there exists a *fractional* H-packing for l using metrics of S, i.e.:

$$\lambda_1 m_1(e) + \cdots + \lambda_N m_N(e) \leq l(e) \qquad \text{for } e \in EG; \tag{27}$$

and

$$\lambda_1 m_1(st) + \cdots + \lambda_N m_N(st) = d_l(st) \qquad \text{for } st \in EH \tag{28}$$

hold for some $\lambda_1, \ldots, \lambda_N \geq 0$. Considering l as above we have from (27) and (28) that

$$c \cdot l \geq \sum_{i=1}^{N} \Big(\lambda_i \sum_{e \in EG} c(e) m_i(e)\Big) \tag{29}$$

and

$$\sum_{st \in EH} g(st) d_l(st) = \sum_{i=1}^{N} \Big(\lambda_i \sum_{e \in EG} g(st) m_i(st)\Big). \tag{30}$$

Comparing (29) and (30) with (26) we obtain the following:

(4.2) *$F(G, H, c, g)$ is solvable iff for any $l: EG \to \mathbb{R}_+$ the inequality*

$$\sum (c(e)m(e): e \in EG) \geq \sum (g(st)m(st): st \in EH) \tag{31}$$

holds for each $m \in S$.

(The 'only if' part of (4.2) follows from (4.1) if we take as l the restriction of $m \in S$ on EG.) Since arguments above can be reversed, we obtain the following relation between H-packings and multicommodity flows, mentioned in the Introduction:

(4.3) *Let G and H be graphs as above, and let S be a set of metrics on VG. Then the following are equivalent*:
(i) *for any c and g, the problem $F(G, H, c, g)$ is solvable iff* (31) *holds for each $m \in S$*;
(ii) *for any $l: EG \to \mathbb{R}_+$, there exists a fractional H-packing for l using metrics in S.*

For example, (4.3) enables us to derive the statement (1.2) directly from the theorem of Papernov [12] (and vice versa): if H is K_4 or C_5 or a union of two stars, then $F(G, H, c, g)$ is solvable iff (31) holds for all cut metrics m on VG. Similarly, a weaker, 'fractional', version of Theorem 1 derives the following fractional version of a theorem in [8]: if $H = K_5$ then $F(G, H, c, g)$ is solvable iff (31) holds for each cut metric and each 2, 3-metric on VG. Note that linear programming duality arguments as above gives relations only between corresponding 'fractional' problems and they, of course, are not sufficient to derive half-integral H-packing theorems, such as Theorem 1 or (1.3), as well as half-integral multicommodity flow theorems, such as in [8] or in [11].

Acknowledgment

I am indebted to the referee who encovered a number of inaccuracies and corrected linguistic errors in the original version of the paper.

References

1. P. Assouad and M. Deza, Metric subspaces of L^1, *Publ. Math. d'Orsay,* **3** (1982), 47 pp.
2. D. Avis, On the extreme rays of the metric cone, *Can. J. Math.*, **32** (1980), 126–144.
3. V. Chvatal, Recognizing intersection patterns, *Ann. Discr. Math.*, **8** (1980), 249–251.
4. M. Deza, On Hamming geometry of unit cubes, *Dokl. Akad. Nauk SSSR,* **134** (1960), 1037–1040 (in Russian).
5. D. Z. Djokovic, Distances preserving subgraphs of hypercubes, *J. Combin. Theory B,* **14** (1973), 263–267.
6. M. Grötschel, L. Lovász and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization,* Springer-Verlag, Berlin–Heidelberg–New York, 1987, 340 pp.
7. A. V. Karzanov, Metrics and undirected cuts, *Math. Programm.*, **32** (1985), 183–198.
8. A. V. Karzanov, Half-integral five-terminus flows, *Discr. Appl. Math.*, **18**(3) (1987), 263–278.
9. A. V. Karzanov, Polyhedra related to undirected multicommodity flows, *Linear Algebra Applics,* **114/115** (1989), 293–328.
10. M. V. Lomonosov, On a system of flows in a network, *Problemy Peredatchi Informacii,* **14** (1978), 60–73 (in Russian).
11. M. V. Lomonosov, Combinatorial approaches to multiflow problems, *Discr. Appl. Math.*, **11**(1) (1985), 1–94.
12. B. A. Papernov, On existence of multicommodity flows, In *Studies in Discrete Optimizations,* A. A. Fridman, ed., Nauka, Moscow, 1976, pp. 230–261 (in Russian).
13. A. Schrijver, Short proofs on multicommodity flows and cuts, To appear.

Received 30 May 1989 and accepted in revised form 15 February 1990

A. V. Karzanov
Institute for System Studies,
Prospect 60 Let Oktyabrya, 9,
117312 Moscow, U.S.S.R.